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LETTER TO THE EDITOR

Energy growth in quantum systems with high dynamical disorder

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Abstract. It is argued that the energy becomes unbounded in time for systems with high dynamical disorder. Consequently the time-evolution could not be periodic or quasiperiodic. Evolution equations are formally equivalent to others found in solid state for systems with static disorder. In this way, it is a surprising result because it is opposite to others known as localization by static disorder where the time-evolution is quasiperiodic. Estimation for the time of relaxation and the diffusion constant are given explicitly. Equivalently our results are also valid at the classical limit with any amount of disorder. A qualitative discussion is carried out in systems with any amount of disorder.

In this letter we consider systems which are perturbed by a random time-dependent external field; namely, we consider systems with *dynamical disorder*. The term is closely related to *static disorder* in solid state physics. Static disorder can be understood by considering materials which contain dissolved impurities at random positions. So static disorder is related to spatially random potentials. The more striking fact relating to static disorder, is that states are spatially localized (Anderson localization). As has been pointed out, localization properties manifest in the fact that the state of the system has a quasiperiodic evolution with time. So, diffusion does not exist in such systems. That statement, about localization, is strictly true in one dimension for any amount of disorder [1]. Moreover, the localization length becomes smaller (high localization) when disorder increases. The situation changes when dynamical disorder is considered. We show that the motion could not be quasiperiodic or periodic, in the regime of high dynamical disorder, in fact, it becomes irreversible. So diffusion, in the sense that the system is not coming to the original state, is found. Surprisingly, our equations of motion are formally equivalent to those found in one-dimensional systems with static disorder where, as we have said, localization exists.

Explicitly, we consider the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\partial_t|\psi\rangle=\left\{\hat{H}_0+V(x)\sum_j f(t-t_j)\right\}|\psi\rangle \tag{1}$$

where the Hamiltonian of the free system has a non-degenerate spectrum E_s defined by

$$\hat{H}_0|s\rangle=E_s|s\rangle \qquad s=1,2,3,\ldots \tag{2}$$

and where we suppose that $E_{s+1}>E_s$. In (1), $\{t_j\}$ is a set of random quantities. We assume that the function $f(t)$ (related to every pulse) is regular, centred at the origin,

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with finite support 2τ and the condition of annulation $f(\pm\tau)=0$ holds. Moreover, superposition does not occur ($t_{j+1}-t_j>2\tau$) and the action of every pulse could be considered independently. So between the two of them, the evolution is given by the free propagator $\hat{G}(t)$ defined as

$$\hat{G}(t)=\exp(-\mathrm{i}\hbar^{-1}\hat{H}_0 t). \tag{3}$$

Assuming that $|t_j+\tau\rangle$ is the out-state just after the action of the j-pulse and $|t_j-\tau\rangle$ the in-state then these two kets are connected by an operator $\hat{K}\hat{G}(2\tau)$, namely

$$|t_j+\tau\rangle=\hat{K}\hat{G}(2\tau)|t_j-\tau\rangle \tag{4}$$

where the post-factor $\hat{G}(2\tau)$, was included for the convenience of future calculations. The above operator describes the action of every pulse on the system while the free propagation is related to the operator $\hat{G}$. We suppose that the collision operator $\hat{K}$ has the following properties:

(*a*) It is unitary because the 'particle' is not missing at the collision.

(*b*) It is independent of t_j because the physical result is not dependent on when the collision is realized.

These two properties are sufficient for our claim. It is evident that the one-cycle evolution, is given by the unitary propagator $\hat{K}\hat{G}(\xi_j)$ or

$$|j+1\rangle=\hat{K}\hat{G}(\xi_j)|j\rangle \qquad \xi_j=t_{j+1}-t_j \tag{5}$$

where the notation $|j\rangle=|t_j+\tau\rangle$ was used and the random independent quantity ξ_j has a j-independent variance, namely

$$\sigma^2=\langle\xi_j^2\rangle-\langle\xi_j\rangle^2. \tag{6}$$

In principle, the asymptotic state of the system could be obtained by iteration of the above relation. It is interesting to note that the random evolution equation (5) is formally equivalent to others found in solid state for one-dimensional spatially disordered systems [2]. In that case, the index j denotes spatial position, and transmission across the sample does not exist as a consequence of localization.

Mean values, for different operators at time j, are related by the density operator $\hat{\rho}^{(j)}$ of the system. Particularly, from (5), the one-cycle evolution equation for this operator is given by

$$\hat{\rho}^{(j+1)}=\hat{K}\hat{G}(\xi_j)\hat{\rho}^{(j)}\hat{G}(-\xi_j)\hat{K}^{-1} \tag{7}$$

which can be written, at energy representation, as

$$\rho_{l,m}^{(j+1)}=\sum_{s,r} J_{s,l}^{*}J_{r,m}\rho_{s,r}^{(j)}\exp(\mathrm{i}\hbar^{-1}\xi_j(E_s-E_r)) \tag{8}$$

where

$$J_{s,r}=\langle s|\hat{K}|r\rangle. \tag{9}$$

Starting from an initial state $|l\rangle$ of $\hat{H}_0$, one can inquire about the probability of finding this state again as $t\to\infty$. Diffusion, of course, occurs since transition is made possible by the hopping term $J_{l,s}$ ($l\neq s$). If the probability of return for $t\to\infty$ is non-zero then we expect that diffusion is restricted to a finite volume in l-space. If, however, the probability of return for $t\to\infty$ goes to zero, the system can diffuse to infinity in l-space. *This last possibility (which we identify as true diffusion) will be realized by* the system in the high disorder regime. It is interesting to note that a similar statement was used by Anderson to define localization in systems with static disorder.

Assuming that the separation between two consecutive levels has a minimum ΔE, namely $|E_s - E_r| > \Delta E$ for any $s \neq r$ then, at the limit of high disorder, the sample averaging of the random phase becomes

$$\langle \exp(\mathrm{i}\hbar^{-1}\xi_j(E_s - E_r)) \rangle \sim \delta_{s,r} \qquad \text{if } \hbar^{-1}\sigma\Delta E \gg 1. \tag{10}$$

If we define the average of diagonal elements of $\hat{\rho}$, as

$$P_l^{(j)} \equiv \langle \rho_{l,l}^{(j)} \rangle \tag{11}$$

we find, from (8) and (10), the equation

$$P_l^{(j+1)} = \sum_s |J_{s,l}|^2 P_s^{(j)} \tag{12}$$

where P_l^j represents the probability that, at time j, the system will be in the state l.

The evolution law (12), has the following important property: *the probability that the system will be at the initial state goes exponentially to zero.* This becomes evident from the fact that if the initial state is $|l\rangle$ then $P_l^j \leqslant |J_{l,l}|^{2j}$ where $|J_{l,l}|^2 < 1$ for any l. So we can define the relaxation time of the system, initially at l-state, $\tau_r(l)$ as

$$\tau_r^{-1}(l) \sim \tau^{-1} \ln\{1/|J_{l,l}|^2\}. \tag{13}$$

The above statement tells us that the evolution, for the random system, could not be periodic or quasiperiodic in time as is the case with static disorder. So diffusion, in the Anderson sense, takes place in the system rapidly. Definition (13), on the relaxation time, is dependent on the initial state; nevertheless a state-independent generalization could easily be made. For instance, choices as $\max\{\tau_r(l)\}$, $1/N\sum_{l=0}^{N}\tau_r(l)$ or eventually others.

At this point a brief comment relating with the classical limit and condition (10), for the annulation of the phase, is necessary. For instance, we consider a particular model defined by the Hamiltonian $\hat{H}_0 = -\partial_\theta^2$, namely the so-called rotator. For this system, the distance between consecutive levels is $\Delta E \sim \hbar^2 l$ $(l = 1, 2, \ldots)$ and their classical limit is related with the condition $l \to \infty$ $(\Delta E/E \sim l^{-1})$ namely, the region of high quantum numbers. In this model the condition of phase-annulation can be written as $\sigma\hbar l \gg 1$ which is fully verified at the limit $l \to \infty$. So a diffusive regime is obtained rapidly in the classical limit for any amount of disorder. This statement can be easily extended to other nonlinear systems where $E_l \sim l^\nu$ $(\nu > 1)$.

To consider explicitly the behaviour of the system, we consider models where the transition matrix $J_{s,r}$ has the form

$$J_{s,r} = \Gamma_{s-r}\, \mathrm{e}^{\mathrm{i}\alpha_{s,r}} \qquad \{\Gamma, \alpha\} \in R \tag{14}$$

then, using the evolution equation (12) for P_l^j, we found that the system has a diffusive behaviour (linear sense) at the l-space, explicitly

$$\sum_l l^2 P_l^j - \left\{\sum_l l P_l^j\right\}^2 = Dj \tag{15}$$

where the positive constant D (the diffusion constant) is given by

$$D=\sum_{l} l^2\Gamma_l^2-\left\{\sum_{l} l\Gamma_l^2\right\}^2. \tag{16}$$

It is easy also to show that, because the average of l has a linear behaviour in time, the relative fluctuations go to zero as $1/\sqrt{j}$, like the behaviour, with the number of particles, for a thermodynamic system in equilibrium.

We note that this diffusive behaviour is not perturbative at the disorder (small disorder). This is interesting because high disorder gives localization at the transport phenomenon in solid state physics whereas here disorder means diffusion in energy-space. Also, we note that, the diffusion equation (12) is not asymptotical in times. Namely, high disorder leads to rapid diffusive behaviour in the system. Proof of diffusion, for a special model (kicked rotator) with any amount of disorder can be found in the paper by Guarneri [3].

If the energy spectrum has, for instance, a point of accumulation (as in the one-dimensional hydrogen atom) then the expression for annulation of the phase (10) cannot be necessarily verified at any position in the spectrum. So, our claim of rapid diffusion at the high disorder regime could be true only locally. For the one-dimensional hydrogen atom in an external electric field, with a particular type of dynamical disorder see, for example [4].

Finally, we consider briefly and qualitatively, the case related to any amount of disorder. The one-cycle evolution operator, for the averaged density operator, in the energy representation (the average of (8)) is related with the unitary transformation $\hat{K}$ and the operator defined by

$$\rho'_{l,s}=\langle\exp(\mathrm{i}\hbar^{-1}\xi(E_l-E_s))\rangle\rho_{l,s} \tag{17}$$

which is a contraction since the average of the phase is less than one. Equality holds if, and only if, $l=s$. Following Guarneri [3], we conclude that there is no non-trivial invariant subspace where the total one-cycle averaged operator acts as a unitary operator. This supports the idea that the energy has a diffusive behaviour (eventually unbounded) for any amount of disorder. In this way, from our results at the high disorder limit, it is not surprising that the system diffuses. For systems with finite dimension N ($\hat{H}_0$, $\hat{K}$ matrices), the invariant subspace of the total one-cycle evolution operator is $\langle\rho_{l,s}\rangle=(1/N)\delta_{l,s}$ which corresponds to the state of maximal entropy.

At this point an important problem is related to how the diffusive regime is developed for any amount of disorder. Perturbation in $1/\sigma$ (first order correction to the high disorder limit) suggests that the evolution equation (12), for the probability P_l^j, is also valid at the limit $j\to\infty$ for any amount of disorder.

On the other hand, it is possible that, because the probability $\rho_{l,l}$ becomes asymptotically zero for any l (equilibrium), the source of the random interaction could be considered as a system of high (eventually infinite) temperature. Moreover, assuming that our model is related to a thermodynamic one, we have not used complex (phenomenological) parameters or non-Hermitian Hamiltonians as many authors use for non-equilibrium problems.

As has been pointed out by the referee, interesting results relating to dissipation in a particular system, the so-called kicked quantum rotator, are close to that presented here. As has been conjectured, dissipation by coupling the momentum coordinate to a heat bath gives energy diffusion in that system [5]. So, the effect of the bath is to destroy localization in momentum space in a similar way to our model but, as we noted, here the coupling is not via momentum coordinate. A study of the kicked quantum rotator with dynamical disorder can be found in [6].

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